

УДК (532.6+539.216.2)

DOI: 10.18384/2310-7251-2018-4-178-190

РЕЗОНАНСНЫЙ ГРАВИТАЦИОННЫЙ ПОТОК СТЕПЕННОЙ ЖИДКОСТИ НАД СКОЛЬЗКОЙ ТОПОГРАФИЧЕСКОЙ ПОДЛОЖКОЙ

Селим Р.С.

*Московский физико-технический институт (государственный университет)
141700, Московская область, г. Долгопрудный, Институтский пер., д. 9, Российская Федерация*

Аннотация. Исследовано гравитационное течение степенной жидкости над скользкой топографической подложкой. Поток жидкости, обусловленный гравитацией, предполагается устойчивым и ограниченным пределом малой амплитуды рифления стенки. В качестве аналитического подхода мы применяем интегральный метод пограничного слоя Кармана–Полхаузена и выводим асимптотическое уравнение, справедливое для довольно тонких плёнок. Полученные результаты подтверждают мнение о том, что резонанс связан с взаимодействием волновой плёнки с капиллярно-гравитационными волнами, движущимися против среднего направления потока. Основным фактором в работе является влияние условия скольжения на резонансные явления при различных степенных показателях n . Исследовано влияние различных параметров на свойства течения и резонансное число Рейнольдса через степенной индекс n .

Ключевые слова: Длинная волна, скользкая волнистая стенка, степенной закон жидкости, линейный резонанс.

A RESONANT GRAVITY-DRIVEN FLOW OF A POWER-LAW FLUID OVER A SLIPPERY TOPOGRAPHY SUBSTRATE

R. Selim

*Moscow Institute of Physics and Technology (State University)
Institutskii per. 9, 141700 Dolgoprudnyi, Moscow region, Russian Federation*

Abstract. The gravity-driven flow of a power-law fluid over a slippery topography substrate is studied. The fluid flow, due to the gravity, is assumed to be steady and confined to the limit of small amplitude of the wall corrugation. As an analytic approach, we apply the Karman–Pohlhausen integral boundary-layer method and derive an asymptotic equation valid for rather thin films. Our results support the view that the resonance is associated with an interaction of the undulated film with capillary-gravity waves travelling against the mean flow direction in the linear case. The influence of the slip condition on the linear resonance phenomena for different power-law indices n is the main factor in this work.

Key words: long wave, slippery wavy wall, power-law fluid, linear resonance.

1. Introduction

Gravity-driven films falling on inclined planes are a fundamental problem in fluid mechanics which has even an exact analytical solution. However, most applications are studied through topography form, related to heat exchangers and coating technologies [22]. Surface rollers, standing waves or hydraulic jumps are new phenomena resulting from the flow over a wavy bottom [23]. Small forces or periodic changes in parameters are the actual reasons for the presence of resonance [10]. Analytical and numerical studies [13, 20] show that the presence of periodic undulation leads to a stable effect which increases with corrugation steepness. The flow and the instability of non-Newtonian films along inclined surfaces were considered by many researchers. Thus, Gupta [6] and Berezin et al. [4] studied the stability of a second-order fluid and a power-law fluid, respectively. Argyriadi et al. [2] found that the extent of distortion depends on the height of corrugations, and so studying the liquid film over a weakly undulating bottom is a simple theoretical technique. The effect of the slip length parameter was examined by Samanta [17], when the solid substrate has a slippery property. Beavers and Joseph [3] discussed the behavior of the interface between a fluid and porous layers in a liquid film which is governed by the Stokes and Darcy equations. Pascal [14] introduced a Navier-slip boundary condition $u = l_s \frac{\partial u}{\partial y}$, where $l_s = \frac{\sqrt{\kappa}}{\beta}$ is the effective slip length

related to the permeability κ and to the empirical dimensionless parameter of Beavers and Joseph. In the framework of the integral boundary layer, the surface waves on a film of a power-law fluid were investigated by Dandapat and Mukhopadhyay [5]. Recently, Amaouche et al. [1] have developed an extension of the model equations derived by Ruyer-Quil and Manneville [16] which correctly predict the linear stability threshold. The porosity of the wavy bottom has an effect on the behavior of a thin liquid film, where a destabilizing influence has been deduced by Thiele et al. [19], especially the existence of a jump boundary condition. The strong effect of resonance occurs at a dimensionless film thickness of about unity [25]. Generation of higher harmonics under the effect of nonlinear resonance results from increasing the wall amplitude and is also found experimentally in a liquid film flowing over sinusoidal substrates [24]. The resonance has a different case between thin and thick films, where the quality of transportation of the bottom perturbation towards the free surface gradually deteriorates in thick films. It remains sharp, but declines in amplitude. Other researchers focused on the influence of surfactants [15] and electric fields [21]. The resonant steady deformation was deduced experimentally by Argyriadi et al. [2], who showed that the dominant characteristic of the unstable free surface only slightly modulates in amplitude and phase during the passage of traveling waves. Saprykin et al. [18] studied inertial effects in the flow of a viscoelastic liquid over a step-down topography. Nevertheless, the influence of the slip length for a non-Newtonian fluid flowing over a wavy wall has not been studied. In this work, we model a non-Newtonian fluid film flowing on an inclined corrugated slippery substrate in the field of steady flow.

The goal of the paper is to examine the linear resonance of the inclined slippery wall with weak periodic corrugations under the effect of the slip length parameter. The paper

is organized as follows. Section 2 addresses the formulation of the problem. In Section 3 we derive the evolution equations of the free surface in the steady state. Section 4 examines the influence of the slippery property on the linear resonance phenomena for different power-law indices n . The summary is given in Section 5.

2. Formulation of the problem

We consider a two-dimensional laminar flow of a thin layer of a power-law fluid flowing down an inclined slippery wavy substrate at inclination angle α . The contour of the topography is given by the periodic function $b(x) = a \cos\left(\frac{2\pi x}{\lambda}\right)$, where x is the coordinate in the mean flow direction, a is the amplitude of the undulation and λ is the wavelength. The film is bounded above by a motionless gas at ambient pressure p_{gas} . The liquid is considered to be non-Newtonian with constant density and surface tension σ_0 . The nonlinear shear-dependent viscosity may be written as

$$\eta = \mu_m \left| \frac{d\gamma}{dt} \right|^{n-1}, \quad (2.1)$$

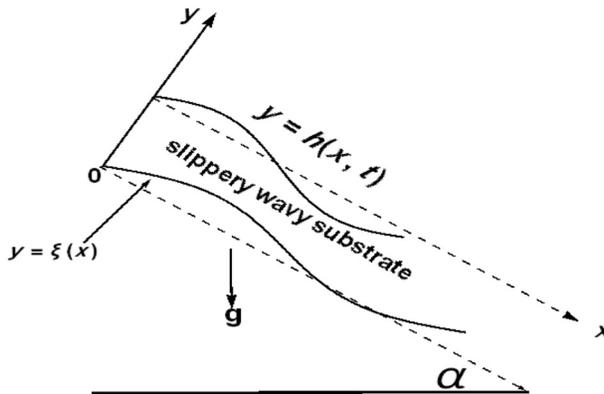


Fig. 1. Sketch of the flow of a power-law fluid over a slippery wavy substrate.

where μ_m is the constant for the particular liquid (μ_m is a measure of the consistency of the fluid; the higher the μ_m , the more viscous the fluid) and n is the positive power-law index. The case $n = 1$ represents a Newtonian fluid with a constant dynamic coefficient of viscosity, while $n < 1$ and $n > 1$ correspond to the case of pseudo-plastic (shear-thinning) and dilatant (shear-thickening) fluids, respectively. We assume that the liquid film is very thin and the induced gravity-driven flow is relatively slow so that the flow regime is close to that predicted by lubrication theory. The adopted (x, y) Cartesian coordinate system is oriented along the main flow direction, which is inclined at an angle α with respect to the horizontal plane, with (x, y) being the stream-wise and cross-stream directions of the flow, respectively.

The continuity and momentum equations that govern the fluid motion have the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2)$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g \sin \alpha + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad (2.3)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} - \rho g \cos \alpha + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}, \quad (2.4)$$

where u and v are the longitudinal and transverse velocity components, respectively, and p is the pressure. Here τ_{ij} is the stress tensor defined by

$$\tau_{ij} = 2\mu_n (2D_{kl}D_{kl})^{\frac{n-1}{2}} D_{ij}, \quad (2.5)$$

where

$$D_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.6)$$

The above equations are subject to the following boundary conditions:

On the wave substrate, the slip condition [11, 12] has the form

$$y = b(x) : |u| = l \left| \frac{\partial u}{\partial y} \right|^n; \quad v = u \frac{\partial b(x)}{\partial x} \quad (2.7)$$

where the constant $l > 0$ is the slip length, and $b(x)$ is the wavy wall substrate.

On the free surface $y = h(x, t)$, we have the following conditions:

$$-p + \left(\left(\frac{\partial h}{\partial x} \right)^2 \tau_{xx} - 2 \left(\frac{\partial h}{\partial x} \right) \tau_{yx} + \tau_{yy} \right) \left(1 + \left(\frac{\partial h}{\partial x} \right)^2 \right)^{-1} + p_0 = \sigma \frac{\partial^2 h}{\partial x^2} \left(1 + \left(\frac{\partial h}{\partial x} \right)^2 \right)^{-\frac{3}{2}}, \quad (2.8)$$

$$\tau_{yx} \left(1 - \left(\frac{\partial h}{\partial x} \right)^2 \right) - (\tau_{xx} - \tau_{yy}) \frac{\partial h}{\partial x} = 0, \quad (2.9)$$

$$v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}, \quad (2.10)$$

The above equations represent the normal, tangential stress balances and the kinematic boundary conditions, respectively, at the free surface interface.

Using the following dimensionless variables [9], we obtain a set of dimensionless parameters that describe the behavior of the model

$$\begin{aligned}
 x^* &= \frac{2\pi x}{\lambda}, \quad y^* = \frac{y}{d}, \quad f^* = \frac{f}{d}, \quad h^* = \frac{h}{d}, \\
 b^* &= \frac{b}{a}, \quad t^* = \frac{2\pi t \bar{u}}{\lambda}, \quad u^* = \frac{u}{\bar{u}}, \quad v^* = \frac{v\lambda}{2\pi d \bar{u}}, \\
 p^* &= \frac{p}{\rho \bar{u}^2}, \quad \tau = \frac{\tau d^n}{\mu_n \bar{u}}, \\
 \bar{u} &= \frac{n}{2n+1} \left(\frac{\rho g \sin \alpha}{\mu_n} \right)^{\frac{1}{n}} d^{\frac{n+1}{n}},
 \end{aligned}$$

where \bar{u} is the mean velocity of the Nusselt film and d is the mean film thickness. After using the above scales, we apply thin-film approximation [8] and then truncate equations and the boundary conditions at order δ^2 , supposing that $\delta \cot \alpha = O(1)$, $\delta \text{Re} = O(1)$, and $\delta \text{Bo}^{-1} = O(1)$. Thus, the dimensionless form of the governing equations and boundary conditions are expressed as follow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.11)$$

$$\delta \text{Re} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\delta \text{Re} \frac{\partial p}{\partial x} + \left(\frac{2n+1}{n} \right)^n + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^n, \quad (2.12)$$

$$0 = -\delta \text{Re} \frac{\partial p}{\partial y} - \delta \cot \alpha \left(\frac{2n+1}{n} \right)^n, \quad (2.13)$$

$$y = \zeta b(x) : |u| = \gamma \left| \frac{\partial u}{\partial y} \right|^n; \quad v = \zeta u \frac{\partial b(x)}{\partial x} \quad (2.14)$$

$$\delta \text{Re} (p - p_{air}) = -\delta \text{Bo}^{-1} \frac{\partial^2 h}{\partial x^2}, \quad (2.15)$$

$$\frac{\partial u}{\partial y} = 0 \quad (2.16)$$

$$v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}, \quad (2.17)$$

where $\text{Bo}^{-1} = \frac{1}{3} \left(\frac{2n+1}{n} \right) \left(\frac{(2\pi l_{ca})^2}{\lambda^2 \sin \alpha} \right)$ is the inverse Bond number with capillary length

$l_{ca} = \sqrt{\frac{\sigma}{\rho g}}$; $\text{Re} = \frac{\rho \bar{u}^{-2-n} d^n}{\mu_n}$ is the Reynolds number; $\delta = \frac{2\pi d}{\lambda}$ is the dimensionless film

thickness parameter that represents a small ratio for thin film thicknesses; $\zeta = \frac{a}{d}$ is the dimensionless steepness parameter; $\gamma = l_s \frac{\bar{n}^{n-1}}{d^n}$ is the dimensionless slip length; components (u, v) are the non-dimensional velocity along x and y directions, respectively; p is the non-dimensional pressure; and $f(x, t) = h(x, t) - \zeta \frac{\partial b}{\partial x}$ is the non-dimensional film thickness.

3. Free Surface Equation

In this Section, we derive the weakly evolution equation of the model and then study the case of a system which is based on limited values for the small parameter $\zeta = \frac{a}{d} < 1$.

Since we have only retained the terms up to the first order of δ , through the latter assumption we integrate Eq. (2.13) with the help of the dynamic normal boundary condition (2.15), where the denominator of the capillary term will be neglected. We obtain the following form of pressure:

$$\delta \operatorname{Re} p(x, y) = \left(\frac{2n+1}{n} \right)^n \delta \cot \alpha (h - y) - 3\delta \operatorname{Bo}^{-1} \frac{\partial^2 h}{\partial x^2}, \quad (3.1)$$

Substituting the right-hand part of equations (3.1) into the stream-wise momentum equation (2.12), we arrive at the expression

$$\begin{aligned} \delta \operatorname{Re} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = & - \left(\frac{2n+1}{n} \right)^n \delta \cot \alpha \frac{\partial h}{\partial x} + \\ & + \left(\frac{2n+1}{n} \right)^n - 3\delta \operatorname{Bo}^{-1} \frac{\partial^3 h}{\partial x^3} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^n, \end{aligned} \quad (3.2)$$

The consistent second-order theory with the Karman–Pohlhausen approximation is supported by experiments in the parametric domain of interest, at least for small amplitude disturbances. Using the Leibniz rule to integrate the partial differential equation (3.2) from the bottom to the free surface, we obtain the evolution equation of the film thickness

$$\begin{aligned} \delta \operatorname{Re} \left(\int_{\zeta b(x)}^{h(x,t)} \frac{\partial u}{\partial t} dy + \int_{\zeta b(x)}^{h(x,t)} u^2 dy \right) = & - \left(\frac{2n+1}{n} \right)^n \delta \cot \alpha f \frac{\partial h}{\partial x} + \left(\frac{2n+1}{n} \right)^n f + \\ & + 3\delta \operatorname{Bo}^{-1} f \frac{\partial^3 h}{\partial x^3} - \left(\frac{\partial u}{\partial y} \right)^n \Big|_{y=\zeta b(x)}. \end{aligned} \quad (3.3)$$

Equation (3.3) is complemented with the dynamic boundary condition (2.16), the steady kinematic condition (2.17), the slip condition at the wavy substrate (2.14), and the continuity equation (2.11). It is to be noted that the momentum integral method

has been used in connection with the boundary layer theory to obtain a single equation for the resonance capillary gravity waves with the bottom corrugation.

In the present application, we exploit the assumption of periodicity and define the domain over only one wall wavelength. This condition creates an additional degree of freedom, which is removed by fixing either the flow rate or the mean liquid level. We select the former and enforce at the inlet the condition

$$Q = \int_{\zeta_b}^h u dy = 1. \tag{3.4}$$

Our study concerns with the concept of steady state that was used by Heining et al. [7] and was confined to the limit of small amplitude of the wall corrugation [13]. To close the system, a specific velocity profile has to be introduced.

$$u = \frac{2n+1}{(n+1)f} \left(1 - \left(1 - \frac{y-\zeta}{f} \right)^{n+1} \right) + \left(\frac{2n+1}{n} \right)^n \gamma f^{-2n}. \tag{3.5}$$

Combining equations (3.4) and (3.5) with Eq. (3.3), we derive the steady evolution equation of the film thickness denoted as the IBL model

$$\begin{aligned} \delta \text{Re} \frac{\partial f}{\partial x} & \left(36\gamma^2 K_2^2 n f^{2-4n} - 9\gamma^2 K_2^2 f^{2-4n} + 12\gamma K_2^2 n f^{1-2n} + \frac{2(2n+1)}{3n+2} \right) + 3K_2 f^{2-2n} \\ & + 3\delta f^3 K_2 \cot \alpha \left(\frac{\partial f}{\partial x} - \zeta \frac{\partial b}{\partial x} \right) - 3f^3 K_2 + 3\delta \text{Bo}^{-1} f^3 \left(\frac{\partial^3 f}{\partial x^3} + \zeta \frac{\partial^3 b}{\partial x^3} \right) = 0, \\ K_2 & = \frac{1}{3} \left(\frac{2n+1}{n} \right)^n. \end{aligned} \tag{3.6}$$

where the subscript represents the differentiation of the film thickness with respect to the variable x . This equation coincides with that in Ref. [9], when the slippery parameter is removed ($\gamma = 0$). Inertia, hydrostatic, capillary pressure, gravity-driven force and wall shear stress are physically responsible for the appearance of the nonlinear terms in Eq. (3.6) governing the film thickness. Let us now express the fluid thickness f as a power series expansion in the form of asymptotic expansion of the parameter $\zeta \ll 1$,

$$f = f_0 + \zeta f_1(x) + O(\zeta^2).$$

By substituting f into equation (3.6), where f_0 is the leading order term, which represents the steady film thickness for a flat incline. A system of linear ordinary differential equations will be degenerated from the nonlinear equation (3.6). At first order we obtain:

$$-3K_2 (2n+1) f_1(x) + (3\delta K_2 \cot \alpha - \delta \text{Re} K_1) \frac{\partial f_1(x)}{\partial x} - 3\delta \text{Bo}^{-1} \frac{\partial^3 f_1(x)}{\partial x^3} = 0, \tag{3.7}$$

where the value of K_2 is formulated above, and the value of K_1 is found to be

$$K_1 = -9\gamma^2 K_2^2 + 36\gamma^2 K_2^2 n + 12\gamma n K_2 + \frac{2(2n+1)}{3n+2}.$$

According to the periodicity of the inhomogeneity, we assume that the solution inherits the periodicity of the substrate and can be written as a periodic function with the same periodicity as the substrate contour; hence,

$$f_1 = A_1 \cos x + B_1 \sin x, \quad (3.8)$$

where the constants A_1 and B_1 have the form

$$B_1 = -\frac{(3K_2(2n+1))(3\delta Bo^{-1} + 3\delta K_2 \cot \alpha)}{9K_2^2(2n+1)^2 + (3\delta Bo^{-1} + 3\delta K_2 \cot \alpha - \delta K_1 \text{Re})^2},$$

$$A_1 = -\frac{(3\delta Bo^{-1} + 3\delta K_2 \cot \alpha)(3\delta Bo^{-1} + 3\delta K_2 \cot \alpha - \delta K_1 \text{Re})}{9K_2^2(2n+1)^2 + (3\delta Bo^{-1} + 3\delta K_2 \cot \alpha - \delta K_1 \text{Re})^2}, \quad (3.9)$$

The position of the free surface up to the first order in the Cartesian coordinates is

$$Y = f_0 + \zeta \sin x + \zeta f_1 = 1 + \zeta a_h \cos(x + \Delta\phi_1) \quad (3.10)$$

with the free-surface amplitude

$$a_h = \sqrt{(1 + B_1)^2 + A_1^2}. \quad (3.11)$$

In order to classify the free surface response at leading order we define the relative free surface amplitude and the relative film thickness amplitude by Eq. (3.11) and by $a_f = \sqrt{B_1^2 + A_1^2}$, respectively.

4. Influence of the slippery property on the resonance phenomenon

In the previous Section, we derived the weakly nonlinear evolution equations for the free surface in the presence of the slippery effect for different power-law indices n . The linear resonance phenomena result from the amplitude amplification of the free surface. The film thickness, the bottom contour and the doubling with each other are the fundamental factors responsible for the presence of inhomogeneity.

In the following figures, the dependences are plotted for $n =$ (solid curve) 0.5, (dotted curve) 1, and (dashed curve) 1.5.

Figure 2(a) shows the amplitude of the free surface up to the first order for different power-law indices in the presence of the slippery property as a function of the Reynolds number. We conclude that the free surface amplitude strongly depends on the power-law index n . At $a_h = 1$, three curves intersect with each other, since the free surface amplitude up to the first order has the same magnitude as the topography amplitude. Then each curve moves to a maximum value that corresponds to a finite value of the resonance Reynolds number. For $n > 1$, the free surface amplitude is smaller than that in the Newtonian case whereas it is higher for $n < 1$. Shear-thinning fluid hence leads

to an amplification of the free surface. It is obvious that a less viscous liquid results in a stronger interaction of the free surface with the topography substrate.

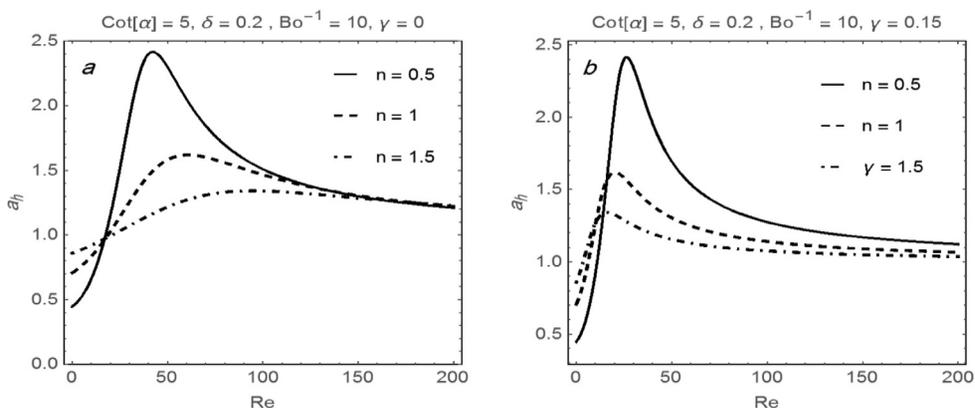


Fig. 2. (a, b) Effect of the slip length on the relative amplitude of the free surface up to the first order for different power-law indices n as a function of the Reynolds number Re , at fixed values of the inverse Bond number $Bo^{-1} = 10$, the dimensionless film thickness $\delta = 0.2$, and the inclination angle $Cot[\alpha] = 5$.

The phenomenon of the surface amplification is called resonance in the literature (see, e.g., [25, 8]), although it is not a dynamic process. It is clear that the behavior of free surface amplitude coincides with that reported in [9], when the wave substrate is free from the property effect.

On the other hand, we have found that the effect of the slip length parameter appears in the value of the resonance Reynolds number according to the power-law index n , as displayed in Fig. 2(b). For $n > 1$, the resonance Reynolds number is close to the Newtonian case, whereas it is higher for $n < 1$. In addition, the values of resonance Reynolds numbers for different power-law indices n during the effect of the slip length parameter are less for $\gamma = 0$.

Figure 3 depicts the amplitude of the film thickness a_f as a function of the Reynolds number. In both cases, it is found that the amplitudes show a maximum, depending on n . The power-law index n has, as already predicted by Fig. 2, a strong influence on both amplitudes. Shear-thickening fluids lead to decreased amplitudes in both cases of the topography substrate, whereas shear-thinning fluids lead to increased amplitudes.

The position of the maximum is referred to as the resonant Reynolds number. For slippery property $\gamma = 0.15$ the resonant Reynolds number decreases with increasing n , whereas the resonant Reynolds number for n on a slippery substrate slowly changes. The curves for non-slip and slip conditions are both related to each other and have the same qualitative shape, as shown in Fig. 3(a, b).

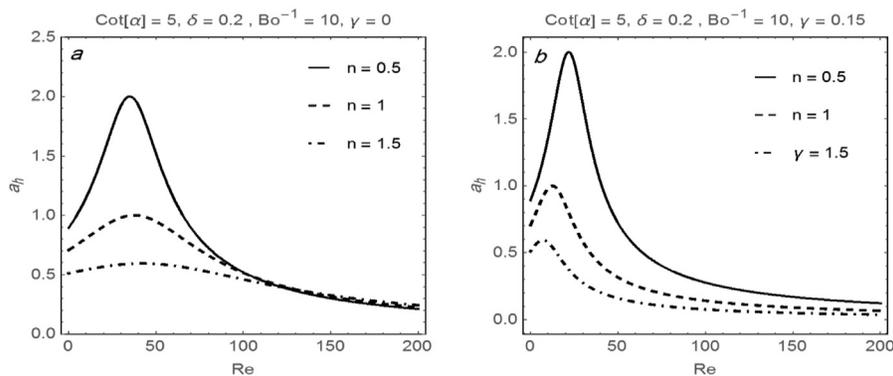


Fig. 3. (a, b) Effect of the slip length on the relative amplitude of the film thickness for different power-law indices n as a function of the Reynolds number Re , at fixed values of the inverse Bond number $Bo^{-1} = 10$, the dimensionless film thickness $\delta = 0.2$, and the inclination angle $Cot[\alpha] = 5$.

5. Conclusion

A linear resonance of the gravity-driven film flow over a slippery topography substrate has been investigated, which extends the study of Heining et al. [9] by incorporating the slip condition at the substrate. We have derived the evolution equation of the free surface amplitude for different power-law indices n under the effect of the slip length parameter. The obtained results describe the effects of slippery property in studying the linear resonance phenomena for small values of the resonance Reynolds number. These results are validated with previous investigations of the resonance phenomenon in gravity-driven films in the absence of slippery property (see, e.g., [25, 7, 9]). The dependence of relative amplitude during a gradual increase in the Reynolds number for both cases of the non-slip and slip length parameter at different power-law indices n indicates that the maximum value for both non-slippery and slippery property are found for a shear-thinning fluid. Moreover, the relative amplitude value at $\gamma = 0.15$ is less than the relative amplitude deduced at $\gamma = 0$ (see Fig. 2).

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ИНФОРМАЦИЯ ОБ АВТОРЕ

Селим Рами Салах Сабер – аспирант кафедры компьютерного моделирования Факультет аэромеханики и летательной техники Московского физико-технического института (государственного университета);
e-mail: selim.rs@phystech.edu.

INFORMATION ABOUT THE AUTHOR

Ramy Salah Saber Selim – post-graduate student at the Department of Computer Modelling, Department of Aeromechanics and Flight Engineering, Moscow Institute of Physics and Technology (State University);
e-mail: selim.rs@phystech.edu.

ПРАВИЛЬНАЯ ССЫЛКА НА СТАТЬЮ

Селим Р.С. Резонансный гравитационный поток степенной жидкости над скользкой топографической подложкой // Вестник Московского государственного областного университета. Серия: Физика-Математика. 2018. № 4. С. 178–190.
DOI: 10.18384/2310-7251-2018-4-178-190

FOR CITATION

Selim R.S. A resonant gravity-driven flow of a power-law fluid over the slippery topography substrate. In: *Bulletin of Moscow Region State University. Series: Physics and Mathematics*, 2018, no. 4, pp. 178–190.
DOI: 10.18384/2310-7251-2018-4-178-190