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TO THE EQUIVALENCE OF THE CATEGORIES OF MODELS AND UNIVERSAL ALGEBRAS

Abstract. *The category equivalence of models and special universal algebras is proved. This result can be used to construct the processing algorithms of different intelligent relation databases. The purpose of this paper is to build the new category called U-ALGEBRAS, which is equivalent to the the MODELS category of the same signature with multivalued homomorphisms as morphisms. The objects of the U-ALGEBRAS category are the same type universal algebras; and morphisms are homomorphisms of universal algebras. The construction has the set-theoretic, algebraic character.*

Key words: *universal algebra, model, intelligent relational databases.*

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К ЭКВИВАЛЕНТНОСТИ КАТЕГОРИЙ МОДЕЛЕЙ И УНИВЕРСАЛЬНЫХ АЛГЕБР

Аннотация. *Доказывается эквивалентность категории моделей и специальной категории универсальных алгебр. Конструкция может быть использована для построения алгоритмов обработки реляционных баз знаний. Целью настоящей заметки является построение категории U-ALGEBRAS, эквивалентной категории моделей одной и той же сигнатуры с многозначными гомоморфизмами в качестве морфизмов. Объектами категории U-ALGEBRAS служат однотипные универсальные алгебры, морфизмами – гомоморфизмы универсальных алгебр. Конструкция носит теоретико-множественный, алгебраический характер.*

Ключевые слова: *универсальная алгебра, модель, реляционные базы знаний.*

The specification of U-ALGEBRAS category makes some interesting aspects, using the topology and fuzzy sets tools. To realize directed search option in any knowledge relational base, you need to select a semantic basis of the primary concepts. A choice of the appropriate topology on the main support of U-ALGEBRAS category is able to help here. That is, it should not to consider all possible subset but only closed in the selected topology. The search time in the one or other relational knowledge base depends essentially on how strong the selected topology is.

The interaction and interrelationship of the system approach, set theory and category theory, binary logic and the theory of algebraic systems, theory of fuzzy sets, multi-valued logic and topology lead to the necessity of constructing a generalized interpretation of the standard model of intellectual knowledge base. This model should be flexible and universal to describe adequately the different subject areas.

Synthetic modelling approach, which consists in the use of various powerful mathematical theories, leads to the formation of new mathematical tools. So, as we see, the concepts of topological model and topological relational system can be successfully used. Indeed, the semantic bases (linguistic thesaurus), hierarchical representation process of basic objects by sets of basic features with the given internal associative connections can be understood and exactly described in the language of topology. With this approach, feature space endowed with the topological structure and a variety of clear and / or fuzzy relations of different arity. State graph obtained by this approach is sufficiently the exact analogue of a simulated situation.

Creation of algorithms for processing relational knowledge base is an actual problem.

Next, we give a sketch of the construction of a joint theory of models and universal algebras.

Definition 1. Model (relational system) $M = \langle A, \Omega \rangle$ is set A with a predetermined set of relations of different arity. The set Ω is called the signature. Arity of relation will indicate superscript in parentheses: $R^{(n)} \in \Omega$.

Definition 2. The correspondence $\varphi \subset A \times B$ called multi-valued (reversible) homomorphism from the model $M_1 = \langle A, \Omega \rangle$ into the model $M_2 = \langle B, \Omega \rangle$ if for any n-ary relation $R^{(n)}$ of signature Ω

$$((a_1, b_1) \in \varphi) \wedge \dots \wedge ((a_n, b_n) \in \varphi) \Rightarrow [R^{(n)}(a_1, \dots, a_n) \Leftrightarrow R^{(n)}(b_1, \dots, b_n)] \quad (1)$$

Models of the same signature with multi-valued homomorphisms as morphisms form a category, which we denote *MODELS*.

Definition 3. Universal algebra $A = \langle A, \Phi \rangle$ is the set A with a specified set of operations Φ of various arity. Under n -ary ($n \geq 1$) operation $\omega^{(n)} \in \Phi$ is meant a mapping $\omega^{(n)}: \underbrace{A \times \dots \times A}_n \rightarrow A$.

0-ary operation fixes a certain element in the set A .

From the definitions 1 and 3 it follows that the universal algebra is a special case of the model. Indeed, for each n -ary operation $\omega^{(n)} \in \Phi$ of universal algebra $A = \langle A, \Phi \rangle$ can be determined $(n+1)$ -ary relation $R^{(n+1)}$ on A by the following formula:

$$R^{(n+1)}(a_1, \dots, a_{n+1}) \Leftrightarrow a_{n+1} = \omega^{(n)}(a_1, \dots, a_n) \quad (2)$$

Definition 4. Mapping $f: A \rightarrow B$ is called homomorphism from universal algebra A into universal algebra B if for each $\omega^{(n)} \in \Phi$ and all $a_1, \dots, a_n \in A$

$$f(\omega^{(n)}(a_1, \dots, a_n)) = \omega^{(n)}(f(a_1), \dots, f(a_n)) \quad (3)$$

Proposition 1. Let $M = \langle A, \Omega \rangle$ is model. For each relation $R^{(n)}$ we define the $(n-1)$ -ary operation on the set 2^A of all subsets A by the following:

a) if $a_1, \dots, a_{n-1} \in A$ then

$$\omega^{(n-1)}(a_1, \dots, a_{n-1}) = \{a \in A : R^{(n)}(a_1, \dots, a_{n-1}, a)\}$$

b) if $M_1, \dots, M_{n-1} \in 2^A$, $M_k \neq \emptyset$, $k = \overline{1, n-1}$, then

$$\omega^{(n-1)}(M_1, \dots, M_{n-1}) = \bigcup_{a_1 \in M_1} \dots \bigcup_{a_{n-1} \in M_{n-1}} \omega^{(n-1)}(a_1, \dots, a_{n-1})$$

c) if at least one of the sets $M_k \in 2^A$, $k = \overline{1, n-1}$ is empty, then

$$\omega^{(n-1)}(M_1, \dots, M_{n-1}) = \emptyset$$

Let Φ be the set of all so defined operations on the set 2^A .

Proof. We define $(n+1)$ -ary relation $R^{(n+1)}$ for each n -ary operation $\omega^{(n)} \in \Phi$ on the set 2^A by following:

$$R^{(n+1)}(M_1, \dots, M_{n+1}) \Leftrightarrow M_{n+1} \subset \omega^{(n)}(M_1, \dots, M_n). \quad (6)$$

Consider the restriction of relation $R^{(n+1)}$ to the set A . Thus, we obtain the model $M = \langle A, \Omega \rangle$ where the signature Ω consists of all so constructed operations $\omega^{(n)} \in \Phi$ or relations $R^{(n+1)}|_A$. According to the Statement 1, we construct the over-model universal algebra $\bar{A} = \langle 2^A, \cup, \emptyset, \tilde{\Phi} \rangle$ and prove that $\bar{A} = A$. To do this, make sure that there is one-to-one correspondence $\phi: \Phi \rightarrow \tilde{\Phi}$ such that for all $\omega^{(n)} \in \Phi$ the equality $\omega^{(n)} = \phi(\omega^{(n)})$ is valid. The correspondence ϕ is constructed as follows: according to formula (3) for any operation $\omega^{(n)} \in \Phi$ there is one-to-one relation $R^{(n+1)} \in \Omega$, and according to positions a) – c) of Proposition 1, for relation $R^{(n+1)}$ there is one-to-one operation $\bar{\omega}^{(n)} \in \tilde{\Phi}$. Let $\phi(\omega^{(n)}) = \bar{\omega}^{(n)}$. If $a_1, \dots, a_n \in A$, then

$$a_{n+1} \in \omega^{(n)}(a_1, \dots, a_n) \Leftrightarrow R^{(n+1)}(a_1, \dots, a_{n+1}) \Leftrightarrow a_{n+1} \in \bar{\omega}^{(n)}(a_1, \dots, a_n).$$

If $M_1, \dots, M_n \in 2^A$, $M_k \neq \emptyset$, $k = \overline{1, n}$, then, according to position b) of Proposition 1,

$$\bar{\omega}^{(n)}(M_1, \dots, M_n) = \bigcup_{a_1 \in M_1} \dots \bigcup_{a_n \in M_n} \bar{\omega}^{(n)}(a_1, \dots, a_n).$$

On the other hand, by identities (5), we obtain

$$\omega^{(n)}(M_1, \dots, M_n) = \bigcup_{a_1 \in M_1} \dots \bigcup_{a_n \in M_n} \omega^{(n)}(a_1, \dots, a_n).$$

Therefore, $\bar{\omega}^{(n)}(M_1, \dots, M_n) = \omega^{(n)}(M_1, \dots, M_n)$.

If at least one of the sets $M_k \in 2^A$, $k = \overline{1, n}$ is empty, then, according to position b) of Statement 1, $\bar{\omega}^{(n)}(M_1, \dots, M_n) = \emptyset$. On the other hand, by identities (4), we obtain $\omega^{(n)}(M_1, \dots, M_n) = \emptyset$. So, we have proved that $\phi(\omega^{(n)}) = \omega^{(n)}$.

Example. Let $M = \langle A, \rho \rangle$ is a model with one binary relation, which is the quasi-order relation, i. e. the following conditions are valid:

$$a \rho a \text{ (reflexivity)} \quad (7)$$

$$(a \rho b) \wedge (b \rho c) \Rightarrow a \rho c \text{ (transitivity)}. \quad (8)$$

In this case, the over model of the universal algebra has the form:

$$A = \langle 2^A, \bigcup, \emptyset, \sigma \rangle,$$

where σ is unary operation on 2^A , satisfying the following identities:

$$\sigma(\emptyset) = \emptyset, \quad (9)$$

$$\sigma\left(\bigcup_{i \in I} M_i\right) = \bigcup_{i \in I} \sigma(M_i) \text{ (linearity identity)}, \quad (10)$$

(I is some set of indexes),

$$M \subset \sigma(M), \quad (11)$$

$$\sigma(\sigma(M)) = \sigma(M) \text{ (idempotency identity)}. \quad (12)$$

Identities (9) and (10) respectively follow identities (4) and (5), the identity (11) and (12) follow directly from (7) and (8).

Note that it follows from the identities (9) – (12), that σ is the Kuratowsky closure operator, with is well known in topology.

It is also easy to see that if in universal algebra with unary operation the identities (9) – (12) are valid, then there is a quasi-order relation ρ on A , such that algebra A is built over the model $M = \langle A, \rho \rangle$.

Proposition 3 follows the Propositions 1 and 2.

Proposition 3. There exists a one-to-one correspondence between models and universal algebras of the form $A = \langle 2^A, \bigcup, \emptyset, \Phi \rangle$, where $\bigcup$ is binary operation of union, $\emptyset$ is 0-ary operation of taking the empty set and for any operation identities (4) and (5) are valid.

Proposition 4. Let $M_1 = \langle A, \Omega \rangle$ and $M_2 = \langle B, \Omega \rangle$ are models with the same signature, $\varphi \subset A \times B$ is a multi-valued homomorphism from M_1 to M_2 . Let $A_1 = \langle 2^A, \bigcup, \emptyset, \Phi \rangle$ and $A_2 = \langle 2^B, \bigcup, \emptyset, \Phi \rangle$ are universal algebra

constructed over the models respectively M_1 and M_2 . Define mapping $f_\varphi : 2^A \rightarrow 2^B$ by follow:

$$\text{a) } f_\varphi(\emptyset) = \emptyset, \quad \text{b) } \forall a \in A \quad f_\varphi(a) = \{b \in B : (a, b) \in \varphi\},$$

$$\text{c) } \forall M \in 2^A \quad f_\varphi(M) = \bigcup_{a \in M} f_\varphi(a).$$

Then the mapping f_φ is a homomorphism of universal algebra A_1 into universal algebra A_2 .

Proof. We need to prove that the mapping has the following properties:

$$f_\varphi(\emptyset) = \emptyset, \tag{13}$$

$$f_\varphi(N \cup M) = f_\varphi(N) \cup f_\varphi(M), \tag{14}$$

$$f_\varphi(\omega^{(n)}(M_1, \dots, M_n)) = \omega^{(n)}(f_\varphi(M_1), \dots, f_\varphi(M_n)) \tag{15}$$

$$\forall \omega^{(n)} \in \Phi, \quad M_1, \dots, M_n \in 2^A.$$

The position a) of proposition 4 follows the property (13). Let's prove the property (14):

$$f_\varphi(N \cup M) = \bigcup_{a \in N \cup M} f_\varphi(a) = \left(\bigcup_{a \in N} f_\varphi(a) \right) \cup \left(\bigcup_{a \in M} f_\varphi(a) \right) = f_\varphi(N) \cup f_\varphi(M).$$

Let's prove the property (15):

$$\begin{aligned} b \in f_\varphi(\omega^{(n)}(M_1, \dots, M_n)) &\Leftrightarrow (\exists a \in \omega^{(n)}(M_1, \dots, M_n) : b \in f_\varphi(a)) \Rightarrow \\ &\Rightarrow (\exists a \in \omega^{(n)}(M_1, \dots, M_n) : (a, b) \in \varphi) \Rightarrow \\ &\Rightarrow (\exists a : R^{(n+1)}(M_1, \dots, M_n, a) \wedge ((a, b) \in \varphi)) \Rightarrow \\ &\Rightarrow (\exists a \in A : \forall a_1 \in M_1, \dots, \forall a_n \in M_n \quad R^{(n+1)}(a_1, \dots, a_n, a) \wedge ((a, b) \in \varphi)) \Rightarrow \\ &\Rightarrow (\forall a_1 \in M_1, \dots, \forall a_n \in M_n : ((a_1, b_1) \in \varphi) \wedge \dots \wedge ((a_n, b_n) \in \varphi) \wedge ((a, b) \in \varphi)) \Rightarrow \\ &\Rightarrow R^{(n+1)}(b_1, \dots, b_n, b) \Leftrightarrow R^{(n+1)}(f_\varphi(M_1), \dots, f_\varphi(M_n), b) \Leftrightarrow \\ &\Leftrightarrow b \in \omega^{(n)}(f_\varphi(M_1), \dots, f_\varphi(M_n)). \end{aligned}$$

Reversing the arguments of proposition 4, we obtain the following result:

Proposition 5. Let $A_1 = \langle 2^A, \cup, \emptyset, \Phi \rangle$ and $A_2 = \langle 2^B, \cup, \emptyset, \Phi \rangle$ are universal algebra constructed over the models $M_1 = \langle A, \Omega \rangle$ and $M_2 = \langle B, \Omega \rangle$. Let $f: A_1 \rightarrow A_2$ be a homomorphism. Then there exists the only multi-valued homomorphism φ from the model M_1 into the model M_2 , so that

$$a) \forall a \in A \quad f(a) = \{b \in B : (a, b) \in \varphi\},$$

$$6) \forall M \in 2^A \quad f(M) = \bigcup_{a \in M} f(a).$$

Definition 6. Category whose objects are universal algebra of the form $A = \langle 2^A, \cup, \emptyset, \Phi \rangle$, with the identities (4) and (5), and whose morphisms are homomorphisms will be called category *U-ALGEBRAS*.

Propositions 1 – 5 lead to the following result.

Theorem. The categories *MODELS* and *U-ALGEBRAS* are equivalent.

We formulate the main conclusions of the construction which is built above.

1) The equivalence of the category of models to some primitive class of universal algebras entails enrichment model theory; in particular, it is possible to transfer to the model concepts congruence, quotient algebra and so on. For example, the concept of congruence leads to a hierarchy of different types of relational systems.

2) The equivalence of the category of models to some category of universal algebra can be a basis for constructing algorithms for processing knowledge relational database.

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